

EQUIVALENCES OF FAMILIES OF STACKY TORIC CALABI-YAU HYPERSURFACES

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ABSTRACT. Given the same anti-canonical linear system on two distinct toric varieties, we provide a derived equivalence between partial crepant resolutions of the corresponding stacky hypersurfaces. The applications include: a derived unification of toric mirror constructions, calculations of Picard lattices for linear systems of quartics in \mathbf{P}^3 , a birational reduction of Reid's list to 81 families, and illustrations of Hodge-theoretic jump loci in toric varieties.

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1. INTRODUCTION

In [DGJ08], Doran, Greene, and Judes described a suite of quintic pencils in finite quotients of \mathbf{P}^4 with identical Picard-Fuchs equations. It was later demonstrated that these pencils are, in fact, all birational [BvGK12] and that the phenomenon occurs, more generally, for all so-called Berglund-Hübsch-Krawitz mirrors [Sho14, Kel13, Cla08].

This birational phenomenon is, perhaps, best viewed as a manifestation of toric geometry, as described in independent work of Borisov, Clarke, and Shoemaker [Bor13, Cla08, Cla14, Sho14]. For example, derived equivalence of the Berglund-Hübsch-Krawitz mirrors to hypersurfaces in the same Gorenstein toric Fano variety [FK14] follows from an examination of the secondary fan associated to the anti-canonical linear system.

In this paper, we recover and generalize some of the results mentioned in [BvGK12, Sho14, Kel13, FK14] with a fairly simple observation. Namely, intersecting polytopes associated to linear systems on toric varieties leads to both birational identifications and derived equivalences among Calabi-Yau Deligne-Mumford stacks.

To set up our main result, let us introduce a bit of notation. Let N and M be dual lattices and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\Sigma \subseteq N_{\mathbb{R}}$ be a complete fan and set ∇_{Σ} to be the convex hull of all minimal generators $u_{\rho} \in N$ of the rays ρ in $\Sigma(1)$. Recall that for a given polytope $\nabla \subset N_{\mathbb{R}}$, we define its polar polytope ∇° as the set

$$\nabla^{\circ} := \{m \in M_{\mathbb{R}} \mid \langle m, n \rangle \geq -1 \text{ for all } n \in \nabla\}.$$

As usual, on a toric variety X_{Σ} , we can use the rays $\rho \in \Sigma(1)$ to enumerate the variables x_{ρ} . Then each lattice point m in ∇_{Σ}° corresponds to a monomial x^m via the correspondence

$$m \longleftrightarrow x^m := \prod_{\rho \in \Sigma(1)} x_{\rho}^{\langle m, u_{\rho} \rangle + 1}.$$

Moreover, $m \in \nabla_{\Sigma}^{\circ}$ if and only if x^m lies in the anti-canonical linear system for the toric variety X_{Σ} . Hence, a set of lattice points $\Xi \subseteq \nabla_{\Sigma}^{\circ} \cap M$ determines an anti-canonical linear system \mathcal{F}_{Ξ} given by the space spanned by the corresponding monomials. Rephrasing, a linear combination of elements in Ξ give a polynomial $\sum c_m x^m$ which corresponds to a hypersurface in X_{Σ} or a stacky hypersurface $\mathcal{Z}_{w, \Sigma}$ in the Cox stack \mathcal{X}_{Σ} (see Definition 2.2).

With this notation in mind, we give our main result:

Theorem 1.1 (= Theorem 2.8). *Let $\Sigma_1, \Sigma_2 \subseteq N_{\mathbb{R}}$ be complete fans and*

$$\Xi \subseteq \nabla_{\Sigma_1}^{\circ} \cap \nabla_{\Sigma_2}^{\circ} \cap M$$

be a finite collection of lattice points inside the intersection of the two corresponding anti-canonical linear systems. Assume that the associated toric varieties $X_{\Sigma_1}, X_{\Sigma_2}$ are projective. Then, for any member w of F_{Ξ} , there exist partial resolutions $\overline{\mathcal{Z}_{w, \Sigma_1}}, \overline{\mathcal{Z}_{w, \Sigma_2}}$ of the corresponding hypersurfaces $\mathcal{Z}_{w, \Sigma_1}, \mathcal{Z}_{w, \Sigma_2}$ in the toric stacks $\mathcal{X}_{\Sigma_1}, \mathcal{X}_{\Sigma_2}$ and a derived equivalence,

$$\mathrm{D}^b(\mathrm{coh} \overline{\mathcal{Z}_{w, \Sigma_1}}) \cong \mathrm{D}^b(\mathrm{coh} \overline{\mathcal{Z}_{w, \Sigma_2}}).$$

Furthermore, if w is not divisible by x_i for some i , then $\mathcal{Z}_{w, \Sigma_1}, \mathcal{Z}_{w, \Sigma_2}$ are birational.

The proof of the theorem is provided in Section 2. In Section 3, we provide five applications. More specifically, the remaining results can be described as follows.

In Subsection 3.1, we discuss when the resolutions are unnecessary and/or when the family is birational and derived equivalent to a hypersurface in a Fano Gorenstein toric variety.

In Subsection 3.2, we examine Reid's list of the 95 weighted-projective 3-spaces where the generic anti-canonical hypersurface has at most Gorenstein singularities. While only fourteen of the weighted-projective spaces correspond to reflexive polytopes, i.e., are Gorenstein toric Fano varieties, each family of K3 surfaces in a weighted-projective 3-space on the list corresponds to the generic family of K3 surfaces on some Gorenstein toric Fano 3-fold. Some of these Gorenstein toric Fano 3-folds are repeated, yielding 81 distinct families after resolving singularities.

In Subsection 3.3, we illustrate how our theorem can be used to realize many families of K3 surfaces as quartic linear systems in \mathbf{P}^3 . This recovers the normal form for a $(E_8 \oplus E_7 \oplus H)$ -polarized family of K3 surfaces in \mathbf{P}^3 discovered independently by Clingher and Doran [CD12] and by Vinberg [Vin13]. Our method, for example, provides 429 ways to realize 52 of the families on Reid's list as families of hypersurfaces in \mathbf{P}^3 up to birational equivalence.

In Subsection 3.4, we demonstrate changes in Hodge numbers for special linear systems in higher dimensions. For example, we give a pencil of quintics in \mathbf{P}^4 which resolves to the mirror quintic pencil. This type of result is in the spirit of Reid's primary observation in

[Rei87] which describes how Calabi-Yau 3-folds can be deformed and resolved in order to alter their discrete invariants.

Finally, in Subsection 3.5, we provide a homological unification of Clarke's mirror construction [Cla08]. This marries many of the mirror constructions in the literature, including, but not limited to, the constructions of Batyrev [Bat94], Berglund-Hübsch-Krawitz [BH93, Kra09], and their generalization by Artebani and Comparin [AC15]. Furthermore, it generalizes the derived equivalence result for Berglund-Hübsch-Krawitz mirrors in [FK14] by dropping a Gorenstein assumption.

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2. DERIVED EQUIVALENCES OF HYPERSURFACES IN LINEAR SYSTEMS

2.1. Background. We work over a field κ of characteristic 0. Let N be a lattice. Consider a fan $\Sigma \subseteq N_{\mathbb{R}}$ with n rays. We can construct a new fan in $\mathbb{R}^{\Sigma(1)}$ given by,

$$\text{Cox}(\Sigma) := \{\text{Cone}(e_{\rho} \mid \rho \in \sigma) \mid \sigma \in \Sigma\}.$$

Enumerating the rays, this fan is a subfan of the standard fan for $\mathbb{A}_{\kappa}^n := \mathbb{A}^n$:

$$\Sigma_t := \{\text{Cone}(e_i \mid i \in I) \mid I \subseteq \{1, \dots, n\}\}.$$

Hence, the associated toric variety $X_{\text{Cox}(\Sigma)}$ over κ is an open subset of \mathbb{A}^n .

Definition 2.1. We call $X_{\text{Cox}(\Sigma)}$ the **Cox open set** associated to Σ .

This quasi-affine variety also comes with the action of a subgroup of the standard torus \mathbb{G}_m^n restricted from its scaling action on \mathbb{A}^n . This is described as follows.

Let M be the dual lattice to N and consider the right exact sequence given by the divisor class map,

$$\begin{aligned} M &\xrightarrow{\text{div}_{\Sigma}} \mathbb{Z}^n \xrightarrow{\pi} \text{Cl}(X_{\Sigma}) \rightarrow 0 \\ m &\mapsto \sum_{\rho \in \Sigma(1)} \langle u_{\rho}, m \rangle e_{\rho} \end{aligned} \tag{2.1}$$

where \mathbb{Z}^n represents the group of torus invariant divisors with basis e_{ρ} and $\text{Cl}(X_{\Sigma})$ is the divisor class group of X_{Σ} .

Applying $\text{Hom}(-, \mathbb{G}_m)$ we get a left exact sequence

$$0 \longrightarrow \text{Hom}(\text{Cl}(X_{\Sigma}), \mathbb{G}_m) \xrightarrow{\hat{\pi}} \mathbb{G}_m^n \xrightarrow{\widehat{\text{div}_{\Sigma}}} \mathbb{G}_m^{\dim M}.$$

We set

$$S_\Sigma := \text{Hom}(\text{Cl}(X_\Sigma), \mathbb{G}_m) \subseteq \mathbb{G}_m^n.$$

Definition 2.2. The Cox stack associated to Σ is the global quotient stack

$$\mathcal{X}_\Sigma := [X_{\text{Cox}(\Sigma)}/S_\Sigma].$$

We can also associate a polytope to our fan

$$\nabla_\Sigma := \text{Conv}(u_\rho | \rho \in \Sigma(1)), \quad (2.2)$$

where u_ρ is the minimal generator for a ray $\rho \in \Sigma(1)$. Recall that any polytope $\nabla \subset N_\mathbb{R}$ has a **polar polytope** ∇° in $M_\mathbb{R}$:

$$\nabla^\circ := \{m \in M_\mathbb{R} | \langle m, n \rangle \geq -1 \text{ for all } n \in \nabla\}.$$

For any finite subset of lattice points

$$\Xi := \{m_1, \dots, m_t\} \subseteq \nabla_\Sigma^\circ \cap M,$$

we get a linear system \mathcal{F}_Ξ of anti-canonical functions on X_Σ defined by

$$w := \sum_{m \in \Xi} c_m x^m.$$

Any such element can be realized both as a hypersurface $Z_{w,\Sigma}$ in the toric variety X_Σ and as a hypersurface $\mathcal{Z}_{w,\Sigma}$ in the stack \mathcal{X}_Σ . The set of all lattice points of ∇_Σ° gives the complete anti-canonical linear system $\mathcal{F}_{\nabla_\Sigma^\circ \cap M}$.

Let us recall the following theorem about derived equivalences amongst such hypersurfaces which comes from varying GIT quotients for the action of S_Σ on \mathbb{A}^n .

Theorem 2.3 (Kawamata, Herbst-Walcher). *Let Σ_1, Σ_2 be simplicial fans with full-dimensional convex support such that*

$$\nabla_{\Sigma_1} = \nabla_{\Sigma_2}.$$

Assume that $X_{\Sigma_1}, X_{\Sigma_2}$ are quasi-projective. Then for any function $w \in \mathcal{F}_\Xi$, as above, there is an equivalence of categories,

$$\text{D}^b(\text{coh } \mathcal{Z}_{w,\Sigma_1}) \cong \text{D}^b(\text{coh } \mathcal{Z}_{w,\Sigma_2}).$$

Proof. If $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ are smooth, this is a toric case of Corollary 4.5 of [Kaw05]. For toric varieties, this can be made entirely explicit using Theorem 2 and Theorem 3 of [HW12]. The result also follows from Theorem 5.2.1 of [BFK12] (version 2 on arXiv) or Cor 4.8 and Prop 5.5 of [H-L12]. For an explicit statement of the theorem in this language, specialize Theorem 1.4 of [FK14] to hypersurfaces. \square

Proposition 2.4. *Let Σ_1, Σ_2 be simplicial fans with full-dimensional convex support such that*

$$\nabla_{\Sigma_1} = \nabla_{\Sigma_2}.$$

Assume that $X_{\Sigma_1}, X_{\Sigma_2}$ are quasi-projective. If $w \in \mathcal{F}_\Xi$ is not divisible by x_i for some i , then $Z_{w,\Sigma_1}, Z_{w,\Sigma_2}$ are birational.

Proof. Both $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ are open substacks of the singular locus of uw in $\mathbb{A}^n \times \text{Spec}(\kappa[u])$ intersected with $Z(u = 0)$. This is nothing more than $Z(w) \subseteq \mathbb{A}^n$. Now, these open substacks are obtained by removing coordinate planes, hence, birationality of $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ follows when any coordinate plane is not an irreducible component. Since these stacks are rigid, this is equivalent to birationality of $Z_{w,\Sigma_1}, Z_{w,\Sigma_2}$ \square

Remark 2.5. In general, the stacks $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ are not isomorphic, e.g., the inertia stacks may differ (see, e.g., Section 5.3 of [Kel13]).

2.2. Blowups and Resolutions. In this section, we compare partial crepant resolutions of toric hypersurfaces by reducing to the situation of Theorem 2.3 and give a proof of the main result Theorem 1.1. First, we give the definition of a star subdivision (called a generalized star subdivision in [CLS11]).

Definition 2.6. Given a fan $\Sigma \subseteq N_{\mathbb{R}}$ and a primitive element $v \in |\Sigma| \cap N$, we define the star subdivision of Σ at v to be the fan $\Sigma^*(v)$ consisting of the following cones

- σ , where $v \notin \sigma \in \Sigma$.
- $\text{Cone}(\tau, v)$, where $v \notin \tau \in \Sigma$ and $\{v\} \cup \tau \subseteq \sigma \in \Sigma$.

Recall that by doing a star subdivision, one obtains an induced projective toric morphism $X_{\Sigma^*(v)} \rightarrow X_{\Sigma}$ (Proposition 11.1.6 of [CLS11]) and that a toric resolution of singularities of X_{Σ} can always be obtained through a sequence of star subdivisions (Theorem 11.1.9 of [CLS11]). We now take a certain sequence of star subdivisions which doesn't necessarily resolve X_{Σ} but produces a desired structure we will later use.

Proposition 2.7. *Let $\Sigma \subseteq N_{\mathbb{R}}$ be a complete fan and*

$$\Delta \subseteq \nabla_{\Sigma}^{\circ}$$

be a full dimensional lattice polytope. Then, there is a simplicial fan Σ^{star} with

$$\nabla_{\Sigma^{\text{star}}} = \text{Conv}(\Delta^{\circ} \cap N)$$

obtained from a sequence of star subdivisions of Σ .

Proof. First take the simplicial refinement of Σ (this is obtained by a sequence of star subdivisions, see Proposition 11.1.7 of [CLS11]). This does not change the rays of Σ hence does not alter ∇_{Σ} . Now, star subdivide the refinement in all the new lattice points in $\text{Conv}(\Delta^{\circ} \cap N)$ and call the resultant fan Σ^{star} . \square

Theorem 2.8. *Let $\Sigma_1, \Sigma_2 \subseteq N_{\mathbb{R}}$ be complete fans and*

$$\Xi \subseteq \nabla_{\Sigma_1}^{\circ} \cap \nabla_{\Sigma_2}^{\circ} \cap M$$

be a finite collection of lattice points inside the intersection of the two corresponding anti-canonical linear systems. Assume that the associated toric varieties $X_{\Sigma_1}, X_{\Sigma_2}$ are projective. Then, for any member w of F_{Ξ} , there exist partial resolutions $\overline{\mathcal{Z}_{w,\Sigma_1}}, \overline{\mathcal{Z}_{w,\Sigma_2}}$ of the corresponding hypersurfaces $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}$ in the toric stacks $\mathcal{X}_{\Sigma_1}, \mathcal{X}_{\Sigma_2}$ and a derived equivalence,

$$\text{D}^b(\text{coh } \overline{\mathcal{Z}_{w,\Sigma_1}}) \cong \text{D}^b(\text{coh } \overline{\mathcal{Z}_{w,\Sigma_2}}).$$

Furthermore, if w is not divisible by x_i for some i , then $Z_{w,\Sigma_1}, Z_{w,\Sigma_2}$ are birational.

Proof. Notice that $\nabla_{\Sigma_1}, \nabla_{\Sigma_2} \subseteq \text{Conv}(\Xi)^\circ$. Furthermore, since they are lattice polytopes, $\nabla_{\Sigma_1}, \nabla_{\Sigma_2} \subseteq \text{Conv}(\text{Conv}(\Xi)^\circ \cap N)$.

By Proposition 2.7, we may blowup $\mathcal{X}_{\Sigma_1}, \mathcal{X}_{\Sigma_2}$ to obtain $\mathcal{X}_{\Sigma_1^{\text{star}}}, \mathcal{X}_{\Sigma_2^{\text{star}}}$ with

$$\text{Conv}(\text{Conv}(\Xi)^\circ \cap N) = \nabla_{\Sigma_1^{\text{star}}} = \nabla_{\Sigma_2^{\text{star}}}.$$

The statement then follows from Theorem 2.3 and Proposition 2.4. \square

Remark 2.9. If $\overline{\mathcal{Z}_{w, \Sigma_1}}$ and $\overline{\mathcal{Z}_{w, \Sigma_2}}$ are connected, then they have trivial canonical bundle by the adjunction formula. Hence, the partial resolutions are crepant.

Corollary 2.10. *Let $\Sigma_1, \Sigma_2 \subseteq N_{\mathbb{R}}$ be complete simplicial fans and*

$$\Xi \subseteq (\nabla_{\Sigma_1}^\circ \cap \nabla_{\Sigma_2}^\circ) \cap M.$$

be any subset. Assume that $X_{\Sigma_1}, X_{\Sigma_2}$ are projective. Suppose that $\mathcal{Z}_{w, \Sigma_1}, \mathcal{Z}_{w, \Sigma_2}$ are smooth hypersurfaces in $\mathcal{X}_{\Sigma_1}, \mathcal{X}_{\Sigma_2}$ defined by the same function,

$$w = \sum_{m \in \Xi} c_m x^m.$$

Then

$$\text{D}^b(\text{coh } \mathcal{Z}_{w, \Sigma_1}) \cong \text{D}^b(\text{coh } \mathcal{Z}_{w, \Sigma_2}).$$

Proof. Since $\mathcal{Z}_1, \mathcal{Z}_2$ are smooth, they are isomorphic to their partial desingularizations. The result then follows from Theorem 2.8. \square

3. APPLICATIONS AND EXAMPLES

3.1. Reflexivity, genericness, and smoothness.

Proposition 3.1 (Bertini, Mullet). *Let X_Σ be a projective toric variety. The generic stacky hypersurface in a basepoint free linear system $\mathcal{Z} \subseteq \mathcal{X}_\Sigma$ is smooth and connected.*

Proof. This is Proposition 6.7 of [Mul09]. \square

Proposition 3.2. *Let $\Sigma \subseteq N_{\mathbb{R}}$ be a simplicial fan such that ∇_Σ is reflexive. Assume X_Σ is projective. Any linear system which contains the vertices of ∇_Σ is basepoint free.*

Proof. Consider the special case where Σ is the normal fan of ∇_Σ° . Then by Proposition 6.1.10 of [CLS11] the anti-canonical linear system is basepoint free. In fact, as in Proposition 6.1.1 of loc. cit., the sections corresponding to the vertices of ∇_Σ° are enough to eliminate base points. The result for a simplicial fan, in general, follows from Theorem 2.3 since smoothness is an invariant of the derived category. \square

Corollary 3.3. *Let $\Sigma_1, \Sigma_2 \subseteq N_{\mathbb{R}}$ be complete fans and*

$$\Xi \subseteq \nabla_{\Sigma_1}^\circ \cap \nabla_{\Sigma_2}^\circ \cap M$$

be a finite set. Assume that $X_{\Sigma_1}, X_{\Sigma_2}$ are projective and that $\text{Conv}(\Xi)$ is reflexive. Then for a generic choice of coefficients

$$w := \sum_{m \in \Xi} c_m x^m$$

there is an equivalence of categories,

$$\text{D}^b(\text{coh } \overline{\mathcal{Z}_{w, \Sigma_1}}) \cong \text{D}^b(\text{coh } \overline{\mathcal{Z}_{w, \Sigma_2}}),$$

where $\overline{\mathcal{Z}_{w,\Sigma_i}}$ is a smooth stacky crepant resolution of the hypersurface defined by w in the stack \mathcal{X}_{Σ_i} . Furthermore, both are derived equivalent to the smooth Calabi-Yau Deligne-Mumford stack $\mathcal{Z}_{w,\Sigma_{\text{Conv}(\Xi)}}$ where $\Sigma_{\text{Conv}(\Xi)}$ is the normal fan to $\text{Conv}(\Xi)$. Moreover, $\mathcal{Z}_{w,\Sigma_1}, \mathcal{Z}_{w,\Sigma_2}, \mathcal{Z}_{w,\Sigma_{\text{Conv}(\Xi)}}$ are all birational.

Proof. This is mostly Theorem 2.8 in the case when the generic member of the partial resolution is smooth and connected which follows from Propositions 3.1 and 3.2. The adjunction formula tells us that $\overline{\mathcal{Z}_{w,\Sigma_1}}, \overline{\mathcal{Z}_{w,\Sigma_2}}$ are smooth Deligne-Mumford stacks with trivial canonical bundle, hence the resolutions are crepant.

Furthermore, the comparisons with $\mathcal{Z}_{w,\Sigma_{\text{Conv}(\Xi)}}$ are immediate as a special case where one of the $\Sigma_i = \Sigma_{\text{Conv}(\Xi)}$. In this case, the generic member is already smooth and connected, again, from Propositions 3.1 and 3.2. \square

We now will give some criteria for certain toric varieties in which their generic stacky anti-canonical hypersurface is smooth.

Definition 3.4. Given a subset

$$I \subseteq \{0, \dots, n\}$$

we say that a monomial p is an I -root if

$$p = \prod_{i \in I} x_i^{e_i}.$$

We say that p is an I -pointer if

$$p = \prod_{i \in I} x_i^{e_i} x_j$$

where $j \notin I$.

Theorem 3.5 (Fletcher). *Let $X_\Sigma = \mathbf{P}(a_0 : \dots : a_n)/G$ where G is a finite abelian group acting multiplicatively. The generic stacky member $\mathcal{Z}_{w,\Sigma}$ of an anti-canonical linear system F_Ξ in X_Σ is smooth if for every nonempty $I \subseteq \{0, \dots, n\}$ there exists an I -root or an I -pointer in Ξ .*

Conversely, suppose $G = 1$ and the generic stacky member $\mathcal{Z}_{w,\Sigma}$ of the complete anti-canonical linear system is smooth. Then, for every nonempty $I \subseteq \{0, \dots, n\}$ there exists an anti-canonical I -root or I -pointer.

Proof. Theorem 8.1 of [Fl00] states the same for the generic member without a finite group action. The proof demonstrates that the sum over these monomials is smooth. Since smoothness is an open property of a linear system, the result follows. Furthermore, since smoothness is a property of polynomials in $\mathbb{A}^{n+1} \setminus 0$, it does not depend on whether or not we quotient by \mathbb{G}_m or $\mathbb{G}_m \times G$. \square

Remark 3.6. One can also compare with similar results obtained by Kreuzer and Skarke. See Theorem 1 of [KS92].

Corollary 3.7. *Suppose X_Σ is not affine and does not contain a torus factor. The generic stacky member $\mathcal{Z}_{w,\Sigma}$ of an anti-canonical linear system F_Ξ in X_Σ is smooth if for every nonempty $I \subseteq \{0, \dots, n\}$ there exists a I -root or an I -pointer in Ξ .*

Conversely, if X_Σ is projective, ∇_Σ is a simplex whose vertices generate N , and the generic stacky member $\mathcal{Z}_{w,\Sigma}$ of the complete anti-canonical linear system is smooth, then, for every nonempty $I \subseteq \{0, \dots, n\}$ there exists an anti-canonical I -root or I -pointer.

Proof. Since X_Σ is not affine and does not contain a torus factor, $X_{\text{Cox}(\Sigma)} \neq \mathbb{A}^{\Sigma(1)}$. Therefore $\text{Cox}(\Sigma)$ is missing at least one cone. Since this is a fan, it must be missing at least the cone $\text{Cone}(e_\rho \mid \rho \in \Sigma(1))$ as all other cones are proper faces. Therefore,

$$X_{\text{Cox}(\Sigma)} \subseteq \mathbb{A}^{\Sigma(1)} \setminus 0.$$

The result the first paragraph then follows from Theorem 3.5 since the stacky member of a linear system on weighted projective space is defined by a function on $\mathbb{A}^{\Sigma(1)} \setminus 0$.

For the converse, let Σ' be the simplicial fan whose rays are the vertices of ∇_Σ . Then, by assumption, $X_{\Sigma'}$ is a weighted projective space. Hence, the converse holds for $X_{\Sigma'}$ by Theorem 3.5. On the other hand, $\nabla_{\Sigma'} = \nabla_\Sigma$ and holds in general by Theorem 2.3 since smoothness is an invariant of the derived category. \square

Corollary 3.8. *Let $\Sigma_1, \Sigma_2 \subseteq N_\mathbb{R}$ be complete fans and*

$$\Xi \subseteq \nabla_{\Sigma_1}^\circ \cap \nabla_{\Sigma_2}^\circ \cap M$$

be a finite set. Assume that $X_{\Sigma_1}, X_{\Sigma_2}$ are projective and that for every nonempty $I \subseteq \{0, \dots, n\}$ there exists a I -root or an I -pointer in Ξ . Then for a generic choice of coefficients

$$w := \sum_{m \in \Xi} c_m x^m$$

there is an equivalence of categories,

$$\text{D}^b(\text{coh } \mathcal{Z}_{w, \Sigma_1}) \cong \text{D}^b(\text{coh } \mathcal{Z}_{w, \Sigma_2}),$$

Moreover, $\mathcal{Z}_{w, \Sigma_1}, \mathcal{Z}_{w, \Sigma_2}$ are birational.

Proof. For a generic choice of coefficients, $\mathcal{Z}_{w, \Sigma_1}, \mathcal{Z}_{w, \Sigma_2}$ are smooth by Corollary 3.7. The result follows from Corollary 2.10. \square

3.2. Equivalence in Reid's List of 95 Families. In [KM12], Kobayashi and Mase find birational correspondences between members of Reid's list of all 95 families of K3 surfaces in weighted-projective space whose generic anti-canonical hypersurface have Gorenstein singularities. These birational correspondences can be recovered by considering the complete linear systems $\nabla_\Sigma^\circ \cap M$ for each X_Σ on Reid's list (see Equation (2.2)).

We will see that there are, in fact, only 81 distinct linear systems. Moreover, by considering the poset of these linear systems under inclusion, one finds that many of the generic families are birational to special families lying in other members of the list.

Corollary 3.9. *Suppose X_Σ is a weighted-projective space on Reid's list (see Table 1). The polytope $\Delta := \text{Conv}(\nabla_\Sigma^\circ \cap M)$ is reflexive and $\mathcal{Z}_{w, \Sigma}$ and $\mathcal{Z}_{w, \Sigma_\Delta}$ are birational if $w \in \mathcal{F}_{\nabla_\Sigma^\circ \cap M}$ is not divisible by x_i for some i , where Σ_Δ is the normal fan to Δ . Hence, generically, the minimal resolutions of $\mathcal{Z}_{w, \Sigma}$ and $\mathcal{Z}_{w, \Sigma_\Delta}$ are isomorphic K3 surfaces. Finally, generically, the stacks $\mathcal{Z}_{w, \Sigma}$ and $\mathcal{Z}_{w, \Sigma_\Delta}$ are smooth and there is an equivalence of categories,*

$$\text{D}^b(\text{coh } \mathcal{Z}_{w, \Sigma}) \cong \text{D}^b(\text{coh } \mathcal{Z}_{w, \Sigma_\Delta}).$$

Proof. Proof of reflexivity and smoothness is exhaustive using Macaulay 2 [GS] and Theorem 3.5.¹ The rest follows directly from Corollary 3.3. \square

¹The Macaulay2 code is available at www.ualberta.ca/~favero/code.html

In the appendix, we give a table giving the weights of X_Σ , the Picard lattice of the resolved generic member of the anti-canonical family (taken from [Bel02]), and the reflexive polytope Δ (indexed by Sage in the `LatticePolytope` package) for all 95 items on Reid's list.

Remark 3.10. There are, in fact, 104 weighted projective 3-spaces such that the generic member of the complete anti-canonical linear system $\mathcal{Z}_{w,\Sigma}$ is smooth. Indeed, in dimension 2 and 3 this is equivalent to $\text{Conv}(\Xi)$ being reflexive where F_Ξ is the complete anti-canonical linear system. In higher dimensions this is no longer true. An excellent summary of the intricate relationship between polytopes and generic smoothness can be found in Section 2.4 of [AC15].

Remark 3.11. The weighted projective space $X_\Sigma = \mathbf{P}(a_1 : a_2 : a_3 : a_4)$ is Gorenstein if and only if a_i divides $\sum a_i$ for all i . This is also equivalent to ∇_Σ being reflexive, in which case $\Delta = \nabla_\Sigma^\circ$. Hence, precisely the first 14 weighted-projective spaces on the list are Gorenstein. The remaining 81 non-Gorenstein weighted-projective spaces have generic anti-canonical hypersurfaces which are birational to generic anti-canonical hypersurfaces in a (different) Gorenstein Fano toric variety. Furthermore, after crepantly resolving, the associated stacks are derived equivalent and generically smooth.

Remark 3.12. There are several cases that yield the same Δ . To be precise, we have the following table:

Reid's Families	index(Δ)
(14), (28), (45), (51)	4080
(20), (59)	3045
(26), (34)	1114
(27), (49)	1949
(38), (77)	3731
(43), (48)	745
(46), (65), (80)	88
(50), (82)	4147
(56), (73)	2
(68), (83), (92)	221

In particular, all such families are point-wise birational and derived equivalent after resolving.

Remark 3.13. Given two weights $a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4)$ in Reid's list, Kobayashi and Mase [KM12] study reflexive subpolytopes $\Delta_{a,b}$ associated to anti-canonical linear systems D_a, D_b in $\mathbf{P}(a), \mathbf{P}(b)$ such that the generic elements Z_a, Z_b of D_a, D_b have the same Picard lattice. In this case, they demonstrate that the resolutions $\overline{Z_a}, \overline{Z_b}$ are isomorphic K3 surfaces. These isomorphisms can all be recovered directly from Theorem 2.8.

3.3. Some Normal Forms in \mathbf{P}^3 . Let X_Σ be a 3-dimensional projective toric variety. If $\Xi \subseteq \nabla_\Sigma^\circ \cap M$ is a finite set and $\text{Conv}(\Xi)$ is reflexive, then, by Corollary 3.3, the generic member $Z_{w,\Sigma}$ of \mathcal{F}_Ξ is birational to $Z_{w,\Sigma_{\text{Conv}(\Xi)}}$ where $\Sigma_{\text{Conv}(\Xi)}$ is the normal fan to $\text{Conv}(\Xi)$. Therefore, the resolutions are isomorphic. In particular, the Picard lattices of the resolutions are isomorphic.

Furthermore, suppose that $\text{Conv}(\Xi)$ is isomorphic to one of the reflexive polytopes Δ in Table 1, i.e., $\text{Conv}(\Xi) = \text{Conv}(\nabla_\Sigma^\circ \cap M)$ where X'_Σ is a weighted projective space appearing Reid's list. By applying Corollary 3.9, we see that $Z_{w,\Sigma}, Z_{w,\Sigma_{\text{Conv}(\Xi)}}$ are also birational to the

corresponding hypersurface $Z_{w,\Sigma'}$ in the weighted projective space $X_{\Sigma'}$ on Reid's list. Hence, for the resolutions, $\overline{Z_{w,\Sigma}}$, $\overline{Z_{w,\Sigma_{\text{Conv}(\Xi)}}}$, $\overline{Z_{w,\Sigma'}}$ we can describe the Picard lattices

$$\text{Pic}(\overline{Z_{w,\Sigma}}) = \text{Pic}(\overline{Z_{w,\Sigma_{\text{Conv}(\Xi)}}}) = \text{Pic}(\overline{Z_{w,\Sigma'}})$$

by invoking Belcastro's list [Bel02]. Belcastro's calculations have been included in Table 1 for the reader's convenience.

Now, consider the special case where Σ is the normal fan to a standard 3-dimensional simplex so that $X_{\Sigma} = \mathbf{P}^3$. In this case, we consider linear systems \mathcal{F}_{Ξ} spanned by quartic monomials in \mathbf{P}^3 . Up to taking convex hull of $\Xi \subseteq \nabla_{\Sigma}^{\circ} \cap M$, there are 429 such choices of linear systems of quartics so that $\text{Conv}(\Xi)$ is isomorphic to some Δ in Table 1. These 429 choices cover exactly 52 of the 95 families in Reid's list and give 44 distinct Picard lattices.²

In conclusion, using Corollary 3.3, we can find 429 linear systems of quartics in \mathbf{P}^3 for which we can describe the Picard lattice of a resolution of the generic member. In total, these linear systems provide 44 distinct Picard lattices. We provide three examples below.

Remark 3.14. Up to taking convex hull, there are 20260 linear systems of quartics in \mathbf{P}^3 such that $\text{Conv}(\Xi)$ is 3-dimensional and reflexive. Exactly 3615 of the 4319 3-dimensional reflexive polytopes occur in such a way.³ The corresponding Picard lattice is described in [Roh04] but nowhere are they listed in terms of the classification of symmetric unimodular lattices in the $K3$ lattice.

Example 3.15. Consider the set

$$\Xi = \{(-1, -1, -1), (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)\}$$

and the corresponding linear system

$$\mathcal{F}_{\mathbf{c}} = c_1x^2yz + c_2xy^2z + c_3xyz^2 + c_4w^4 + c_5xyzw$$

of quartics in \mathbf{P}^3 .

Then

$$\text{Conv}(\Xi) = \Delta_0$$

is the reflexive polytope of index 0. Hence, a resolution of a generic member of this family is polarized by $(E_8 \oplus E_8 \oplus \langle -4 \rangle \oplus H)$ since this appears as Reid Family 52 (See Table 1).

Example 3.16. Consider the set

$$\Xi = \{(-1, -1, -1), (-1, -1, 0), (0, 0, 0), (-1, 1, 0), (2, -1, 0), (-1, -1, 1)\}$$

and the corresponding linear system

$$\mathcal{F}_{\mathbf{c}} = c_1w^4 + c_2w^3z + c_3xyzw + c_4y^2wz + c_5x^3z + c_6w^2z^2$$

of quartics in \mathbf{P}^3 .

Then

$$\text{Conv}(\Xi) = \Delta_{88}$$

is the reflexive polytope of index 88. Hence, a resolution of a generic member of this family is polarized by $(E_8 \oplus E_8 \oplus H)$ since this appears as Reid Families 46, 65, and 80 (See Table 1).

²The Macaulay2 code and full lists can be found at www.ualberta.ca/~favero/code.html

³The Sage code and full lists can be found at www.ualberta.ca/~favero/code.html

Example 3.17. In this example we produce a family of $(E_8 \oplus E_7 \oplus H)$ -polarized K3 surfaces in a blow-up of \mathbf{P}^3 . This was achieved independently by Clingher and Doran [CD12] and Vinberg [Vin13] using non-toric methods.

Consider the set

$$\Xi = \{(-1, -1, -1), (-1, -1, 0), (0, 0, 0), (-1, 1, 0), (2, -1, 0), (-1, -1, 1), (0, -1, 1)\}$$

and the corresponding linear system

$$\mathcal{F}_c = c_1 w^4 + c_2 w^3 z + c_3 x y z w + c_4 y^2 w z + c_5 x^3 z + c_6 w^2 z^2 + c_7 x z^2 w$$

of quartics in \mathbf{P}^3 .

Then

$$\text{Conv}(\Xi) = \Delta_{221}$$

is the reflexive polytope of index 221. Hence, a resolution of a generic member of this family is polarized by $(E_8 \oplus E_7 \oplus H)$ since this appears as Reid Families 68, 83, and 92 (See Table 1).

Remark 3.18. Take X_Σ to be a complete, simplicial toric variety. A theorem of Bruzzo and Grassi (Theorem 3.8 of [BG12]) states that, given the family \mathcal{F} of anticanonical hypersurfaces in a toric variety X_Σ , the Picard rank of a very general hypersurface $X \in \mathcal{F}$ is the Picard rank of X_Σ . We define the **Noether-Lefschetz locus** to be the subvariety $\text{NL}(X_\Sigma)$ in \mathcal{F} where $X \in \text{NL}(X_\Sigma)$ has higher Picard rank than that of X_Σ .

While, we can give many examples of subloci of the ambient space locus using the technique above, the hypersurfaces in our subloci are typically not smooth. So, while the resolutions are of higher picard rank, the hypersurfaces themselves are not, a priori, in the Noether-Lefschetz locus. A paper by Bruzzo and Grassi on Noether-Lefschetz loci in these toric varieties is in preparation [BG].

3.4. Cohomological Jump Loci in families of hypersurfaces in toric varieties. In this subsection, we will discuss special families of Calabi-Yau hypersurfaces in a toric variety whose resolutions do not have the same Hodge numbers as the generic Calabi-Yau hypersurface in the toric variety. First, using work of Popa [Po13], we have the following consequence of Corollary 3.3.

Corollary 3.19. *With the same notation and assumptions of Corollary 3.3, we have an isomorphism of sums of Hodge groups,*

$$\bigoplus_{p-q=i} H_{\text{orb}}^{p,q}(\overline{\mathcal{Z}_{w,\Sigma_1}}) \cong \bigoplus_{p-q=i} H_{\text{orb}}^{p,q}(\overline{\mathcal{Z}_{w,\Sigma_2}})$$

for every i . Assume in addition that $\overline{\mathcal{Z}_{w,\Sigma_1}}, \overline{\mathcal{Z}_{w,\Sigma_2}}$ are Gorenstein of dimension up to three. Then we get an equality of Hodge numbers,

$$h^{p,q}(\overline{\mathcal{Z}_{w,\Sigma_1}}) = h^{p,q}(\overline{\mathcal{Z}_{w,\Sigma_2}}).$$

Proof. This follows immediately from Corollary 3.3 and Theorem A and Corollary C of [Po13]. \square

Example 3.20. Corollary 3.19 can be used to revisit Example 3.15 in a higher dimensional context. Let $\{e_1, \dots, e_n\}$ be a basis for N and $\{e_1^*, \dots, e_n^*\}$ be the dual basis of M . Let

$$\Xi = \{e_1^*, \dots, e_n^*, -\sum e_i^*\}.$$

Then, $\text{Conv}(\Xi)$ is the polar dual to a standard simplex of side length $n+1$, hence reflexive.

The linear system \mathcal{F}_Ξ , after rescaling by the torus, can be reduced to a one-parameter family of polynomials,

$$\mathcal{F}_\psi := \sum_{i=0}^n x_i^{n+1} - \psi \prod_{i=0}^n x_i.$$

This gives a pencil of (stacky) hypersurfaces $\mathcal{X}_\psi = Z(\mathcal{F}_\psi) \subset [\mathbf{P}^n / (\mathbb{Z}_{n+1})^{n-1}]$. Similarly, the one-parameter family of polynomials,

$$G_\psi := \left(\sum_{i=0}^{n-1} x_i \right) \left(\prod_{i=0}^{n-1} x_i \right) + x_n^{n+1} - \psi \prod_{i=0}^n x_i$$

gives a family of hypersurfaces $\mathcal{Y}_\psi = Z(G_\psi) \subset \mathbf{P}^n$. By Corollary 3.3 we have,

$$\text{D}^b(\text{coh } \mathcal{X}_\psi) = \text{D}^b(\text{coh } \mathcal{Y}_\psi).$$

Hence, for generic ψ , by Corollary 3.19, we have

$$\bigoplus_{p-q=i} \text{H}_{\text{orb}}^{p,q}(\mathcal{X}_\psi) \cong \bigoplus_{p-q=i} \text{H}_{\text{orb}}^{p,q}(\overline{\mathcal{Y}_\psi})$$

for all i . Furthermore, since both families are Gorenstein, when $n \leq 4$, we get

$$h^{p,q}(\mathcal{X}_\psi) = h^{p,q}(\overline{\mathcal{Y}_\psi})$$

for all p, q .

Further specializing to the case where $n = 4$, for generic ψ , the hypersurface \mathcal{X}_ψ has Hodge numbers $(h^{2,1}, h^{1,1}) = (1, 101)$, so the crepant resolution of the corresponding quintic $\overline{\mathcal{Y}_\psi}$ in \mathbf{P}^4 must also.

Remark 3.21. The family of quintics G_ψ is a singular realization of the Batyrev mirror quintic family inside (a different) \mathbf{P}^4 . Indeed, a generic hypersurface in \mathbf{P}^4 has the mirror Hodge numbers $(h^{2,1}, h^{1,1}) = (101, 1)$.

Remark 3.22. More generally, let Δ, Δ' be reflexive polytopes and suppose you have a strict containment $\Delta \subseteq \Delta'$. Then the set of all lattice points in Δ gives a linear system on $\mathcal{X}_{\Sigma_{\Delta'}}$. After taking crepant resolutions, the members of this linear system have different Hodge numbers and in fact different Euler characteristics than the members of the generic linear system corresponding to the lattice points of Δ . There are quite a few examples, as there are 473,800,776 reflexive 4-dimensional polyhedra which, for example, must lie in one of the 308 maximal reflexive polytopes (see Section 4 of [KS00]).

3.5. Equivalences of Exotic Mirror Constructions. Given a Calabi-Yau hypersurface in a toric variety, there could possibly be many different constructions for a conjectural mirror. The most standard mirror construction is that predicted by Batyrev that states the family of Calabi-Yau hypersurfaces in \mathbf{P}_Δ is mirror to the family of Calabi-Yau hypersurfaces in $\mathbf{P}_{\Delta^\circ}$ [Bat94], where \mathbf{P}_Δ is the toric variety X_Σ where Σ is the normal fan to Δ . In this section, we will provide a few alternative mirror constructions and prove that they all agree with Batyrev's construction up to resolution and derived equivalence.

3.5.1. *Clarke Mirror Symmetry.* As always, M and N are dual lattices and $\Sigma \subseteq N_{\mathbb{R}}$ a fan. We work over the field $\kappa = \mathbb{C}$. Recall from Equation (2.1) that there is a right exact sequence:

$$\begin{aligned} M &\xrightarrow{\text{div}_{\Sigma}} \mathbb{Z}^n \xrightarrow{\pi} \text{Cl}(X_{\Sigma}) \rightarrow 0 \\ m &\mapsto \sum_{\rho \in \Sigma(1)} \langle u_{\rho}, m \rangle e_{\rho}. \end{aligned} \quad (3.1)$$

Now consider a finite set

$$\Xi \subseteq \nabla_{\Sigma}^{\circ} \cap M.$$

This gives another right exact sequence,

$$\begin{aligned} N &\xrightarrow{\text{mon}_{\Xi}} \mathbb{Z}^{\Xi} \xrightarrow{q} \text{coker}(\text{mon}_{\Xi}) \rightarrow 0 \\ n &\mapsto \sum_{m \in \Xi} \langle m, n \rangle e_m. \end{aligned} \quad (3.2)$$

The realization of X_{Σ} as a GIT quotient amounts to a choice of a S_{Σ} -linearization of $\mathcal{O}_{\mathbb{A}^n}$ or, equivalently, an element of $D \in \text{Cl}(X_{\Sigma})$. Indeed, the fan Σ can be recovered from (3.1) and D as the normal fan to the polytope $(\pi \otimes_{\mathbb{Z}} \mathbb{R})^{-1}(D \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{R}_{\geq 0}^n$.

Similarly, a specific hypersurface with nonzero monomial coefficients in \mathcal{F}_{Ξ} is a choice of coefficients $\mathbf{c} \in (\mathbb{C}^*)^{\Xi}$ which, up to reparametrization by the torus $N \otimes_{\mathbb{Z}} (\mathbb{C}^*)^{\Xi}$, amounts to a choice $q(\mathbf{c}) \in \text{coker}(\text{mon}_{\Xi}) \otimes_{\mathbb{Z}} \mathbb{C}$. Here, the \mathbb{Z} -module structure on \mathbb{C} is the multiplicative structure.

Clarke's mirror construction simply exchanges the roles of the two right exact sequences above and the choices $D \otimes_{\mathbb{Z}} \mathbb{C}^*$, $q(\mathbf{c})$. More precisely, given X_{Σ} and an equivalence class $q(\mathbf{c})$ in a linear system Ξ , we get two stacks,

$$\mathcal{Z}_{q(\mathbf{c}), \Sigma}, \mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi, q(\mathbf{c})}},$$

where $\Sigma_{\Xi, q(\mathbf{c})}$ is the fan associated to (3.2) and $q(\mathbf{c})$, i.e., the normal fan to the polytope $(q \otimes_{\mathbb{Z}} \mathbb{R})^{-1}(\text{im } q(\mathbf{c})) \cap \mathbb{R}_{\geq 0}^{\Xi}$.

Definition 3.23 (Clarke [Cla08]). The stacks

$$\mathcal{Z}_{q(\mathbf{c}), \Sigma}, \mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi, q(\mathbf{c})}}$$

are called *mirror* to one another.

Remark 3.24. To complete the symmetry, one can more generally choose the right exact sequence (3.1) and an element $\bar{D} \in \text{Cl}(X_{\Sigma}) \otimes_{\mathbb{Z}} \mathbb{C}$. This amounts to a choice of B -field on X_{Σ} [Cla08].

Remark 3.25. Since we only choose an equivalence class of the defining polynomial w , the stacks above are only defined as subsets of $\mathcal{X}_{\Sigma}, \mathcal{X}_{\Sigma_{\Xi}}$ up to reparametrization by $\mathbb{G}_m^{\Sigma(1)}, \mathbb{G}_m^{\Xi}$.

Notice that if we choose different anti-canonical linear systems Ξ_1, Ξ_2 and nonzero coefficients

$$\mathbf{c}_i : \Xi_i \rightarrow \mathbb{C}^*$$

then we can get two smooth stacky Calabi-Yau hypersurfaces $\mathcal{Z}_{q(\mathbf{c}_1), \Sigma}$ and $\mathcal{Z}_{q(\mathbf{c}_2), \Sigma}$ in \mathcal{X}_{Σ} . While this varies only the complex structure of the Calabi-Yau hypersurface, the mirrors will lie in different toric varieties. On the other hand, such a variation should only affect the symplectic structure of the mirror and not the complex structure. This is explained by the following result which says that the mirror “B-models” are the same.

Corollary 3.26. *Let $\Sigma \subseteq N_{\mathbb{R}}$ be a complete fan and*

$$\Xi_1, \Xi_2 \subseteq \nabla_{\Sigma}^{\circ}$$

be finite sets. For any choice of coefficients

$$\mathbf{c}_i : \Xi_i \rightarrow \mathbb{C}^*$$

consider the stacky hypersurfaces $\mathcal{Z}_{q(\mathbf{c}_1), \Sigma}, \mathcal{Z}_{q(\mathbf{c}_2), \Sigma} \subseteq \mathcal{X}_{\Sigma}$. Then the two corresponding mirror coarse moduli spaces $\overline{\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^, \Sigma_{\Xi_1, q(\mathbf{c}_1)}}}, \overline{\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_2, q(\mathbf{c}_2)}}}$ are birational. Moreover, the partial resolutions $\overline{\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_1, q(\mathbf{c}_1)}}}$ and $\overline{\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_2, q(\mathbf{c}_2)}}}$ are derived equivalent.*

Proof. The mirror stacks,

$$\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_1, q(\mathbf{c}_1)}}, \mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_2, q(\mathbf{c}_2)}},$$

are anti-canonical members of $F_{\nabla_{\Sigma} \cap N}$ in $X_{\Sigma_{\Xi_1, q(\mathbf{c}_1)}}, X_{\Sigma_{\Xi_2, q(\mathbf{c}_2)}}$. Now notice, from the exact sequence (3.2), that the rays of $\nabla_{\Sigma_{\Xi_i, q(\mathbf{c}_i)}}$ are nothing more than the elements of Ξ_i . Therefore,

$$\nabla_{\Sigma_{\Xi_1, q(\mathbf{c}_1)}} = \text{Conv}(\Xi_1) \subseteq M_{\mathbb{R}} \text{ and } \nabla_{\Sigma_{\Xi_2, q(\mathbf{c}_2)}} = \text{Conv}(\Xi_2) \subseteq M_{\mathbb{R}}.$$

Hence,

$$\nabla_{\Sigma} \subseteq \nabla_{\Sigma_{\Xi_1, q(\mathbf{c}_1)}}^{\circ} \cap \nabla_{\Sigma_{\Xi_2, q(\mathbf{c}_2)}}^{\circ} \subseteq N_{\mathbb{R}}.$$

Therefore, $\mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_1, q(\mathbf{c}_1)}}, \mathcal{Z}_{D \otimes_{\mathbb{Z}} \mathbb{C}^*, \Sigma_{\Xi_2, q(\mathbf{c}_2)}}$ satisfy the hypotheses of Theorem 2.8. \square

Remark 3.27. The birational aspect of this result was first proven by Artebani and Comarin as Proposition 3.5 in [AC15] in a slightly more restrictive context; a paper in which they give a polytope-theoretic approach to the unification of Berglund-Hübsch-Krawitz and Batyrev mirrors. A derived equivalence analogue to their Proposition 3.5 is directly implied by the above corollary.

Remark 3.28. In the upcoming work of Aspinwall and Plesser [AP], they provide a general mirror construction for Landau-Ginzburg models associated to two Gorenstein cones which are contained in one another's duals. In their theory, they allow non-geometric phases as gauged linear sigma models. Comparing these phases relates to an algebraic condition which inspired our results, see Section 4 and especially Corollary 4.7 of [FK14]. The method of the current paper is a toric description which focuses only on geometric phases related to Gorenstein cones of index 1.

3.5.2. Berglund-Hübsch-Krawitz Mirror Symmetry. A special but important case of Clarke's construction was provided by Berglund and Hübsch [BH93]. It works over an arbitrary field κ and goes as follows.

We consider a finite quotient of weighted projective space $X_{\Sigma} = \mathbf{P}(q_0 : \dots : q_n)/G$ and a G -invariant weighted homogeneous polynomial which is a sum of $n+1$ monomial terms

$$w_A := \sum_{i=0}^n \prod_{j=0}^n x_j^{a_{ij}}. \quad (3.3)$$

This can be encoded in a matrix A with entries $(a_{ij})_{i,j=0}^n$. This gives a linear system F_{Ξ} where

$$\Xi_A = \{m \in M \mid x^m = \prod_{j=0}^n x_j^{a_{ij}} \text{ for some } j\}.$$

Now, assume that A is invertible and that $\mathcal{Z}_{w_A, \Sigma}$ is smooth and Calabi-Yau. It turns out that $X_{\Sigma_{\Xi_A, w_A}} = \mathbf{P}(q_0 : \dots : q_n)/H$ where the group H is the so-called “dual group” introduced

by Krawitz [Kra09]. Furthermore, the function determined by $D \otimes_{\mathbb{Z}} \mathbb{C}$ corresponds to the polynomial associated to the transpose matrix w_{AT} . Good references are Clarke's original paper [Cla08], the toric reinterpretations of Berglund-Hübsch-Krawitz mirror symmetry due to Borisov [Bor13] and Shoemaker [Sho14], or the summary provided in [FK14].

Corollary 3.29. *Given two stacky Calabi-Yau hypersurfaces $\mathcal{Z}_{w_A, \Sigma}$ and $\mathcal{Z}_{w_{A'}, \Sigma}$, the Berglund-Hübsch-Krawitz mirrors $\mathcal{Z}_{w_{AT}, \Sigma_{\Xi_A, w_A}}$ and $\mathcal{Z}_{w_{AT}, \Sigma_{\Xi_{A'}, w_{A'}}}$ are birational and derived equivalent.*

Proof. This is a special case of Corollary 3.26 since the stacks $\mathcal{Z}_{w_{AT}, \Sigma_{\Xi_A, w_A}}$ and $\mathcal{Z}_{w_{AT}, \Sigma_{\Xi_{A'}, w_{A'}}}$ are smooth [KS92]. In fact, it works over an arbitrary algebraically closed field of characteristic 0 since the coefficients are in \mathbb{Z} . Birationality of the coarse moduli space is the same as birationality of the stacks since they are rigid. \square

Remark 3.30. The birational aspect of the corollary above was first proven by Shoemaker in [Sho14] and the derived aspect was proven in [FK14]. The latter assumes that X_{Σ} is Gorenstein. The result above drops the Gorenstein assumption.

APPENDIX A. REID'S LIST OF 95 WEIGHTED-PROJECTIVE SPACES

We now provide a table giving the weights of X_{Σ} , the Picard lattice of the resolved generic member of the anti-canonical family (taken from [Bel02]), and the reflexive polytope Δ (indexed by Sage in the `LatticePolytope` package) for all 95 items on Reid's list.

No.	weights	Picard Lattice	index(Δ)
1	(1,1,1,1)	$\langle 4 \rangle$	4311
2	(2,3,3,4)	$E_6 \perp D_4 \perp U(3)$	1945
3	(1,1,2,2)	$M_{(1,1,1),(1,1,1),0}$	4281
4	(1,3,4,4)	$T_{4,4,4}$	3727
5	(1,1,1,3)	$\langle 2 \rangle$	4317
6	(1,2,2,5)	$D_4 \perp U(2)$	4255
7	(1,1,2,4)	$M_{(1,1),(1,1),0}$	4312
8	(1,2,3,6)	$M_{(1,1,2,2),(1,1,1,1),-2}$	4228
9	(1, 4, 5, 10)	$T_{2,5,5}$	3993
10	(1,1, 4, 6)	U	4318
11	(2, 3, 10, 15)	$E_6 \perp D_4 \perp U$	3038
12	(1,2,9,6)	$D_4 \perp U$	4282
13	(1,3,8,12)	$E_6 \perp U$	4229
14	(1, 6, 14, 21)	$E_8 \perp U$	4080
15	(3,3,4,5)	$E_6 \perp (A_2)^3 \perp U$	751
16	(3,6,7,8)	$E_8 \perp (A_2)^3 \perp U$	87
17	(2,3,5,5)	$T_{2,5,5} \perp A_4$	1538
18	(1,2,3,3)	$M_{(1,2,2,2),(1,1,1,1),-2}$	4005
19	(1,2,2,3)	$M_{(1,1,1,1,2),(1,1,1,1,1),-2}$	4091
20	(1,6,8,9)	$E_8 \perp A_2 \perp U$	3045
21	(1,1,1,2)	$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$	4309
22	(1,3,5,6)	$E_6 \perp A_2 \perp U$	3733
23	(2,2,3,5)	$D_5 \perp D_4 \perp U(2)$	2705

No.	weights	Picard Lattice	index(Δ)
24	(1,2,4,5)	$D_4 \perp A_2 \perp U$	4083
25	(1,1,3,4)	$A_2 \perp U$	4308
26	(2,4,5,9)	$D_8 \perp D_4 \perp U$	1114
27	(2,3,8,11)	$E_8 \perp D_4 \perp U$	1949
28	(1,3,7,10)	$E_8 \perp U$	4080
29	(4,5,6,15)	$T_{2,5,5} \perp D_6$	223
30	(5,7,8,20)	$E_8 \perp T_{2,5,5}$	31
31	(3,4,5,12)	$E_6 \perp A_7 \perp U$	749
32	(2,2,3,7)	$D_4 \perp D_4 \perp U(2)$	3322
33	(2,3,4,9)	$M_{(1,1,1,1,2,2,3),(1,1,1,1,1,1),-4}$	2356
34	(2,6,7,15)	$D_8 \perp D_4 \perp U$	1114
35	(3,4,7,14)	$E_8 \perp A_6 \perp U$	760
36	(2,3,5,10)	$T_{2,5,5} \perp A_3$	2359
37	(1,3,4,8)	$T_{3,4,4}$	4088
38	(1,3,5,9)	$E_8 \perp A_1 \perp U$	3731
39	(1,6,8,15)	$E_6 \perp A_1 \perp U$	4082
40	(1,2,4,7)	$D_4 \perp A_1 \perp U$	4233
41	(2,3,7,12)	$E_6 \perp D_5 \perp U$	2357
42	(1,1,3,5)	$A_1 \perp U$	4316
43	(3,4,11,18)	$E_8 \perp E_6 \perp U$	745
44	(1,2,5,8)	$D_5 \perp U$	4256
45	(1,4,9,14)	$E_8 \perp U$	4080
46	(5,6,22,33)	$E_8^2 \perp U$	88
47	(3,4,14,21)	$E_7 \perp E_6 \perp U$	1124
48	(3,5,16,24)	$E_8 \perp E_6 \perp U$	745
49	(2,5,14,21)	$E_8 \perp D_4 \perp U$	1949
50	(1,4,10,15)	$E_7 \perp U$	4147
51	(1,5,12,18)	$E_8 \perp U$	4080
52	(7,8,9,12)	$(E_8)^2 \perp \langle -4 \rangle \perp U$	0
53	(3,4,5,6)	$M_{(1,2,2,2,3,4),(1,1,1,1,1,1),-4}$	246
54	(3,5,6,7)	$E_8 \perp (A_2)^3 \perp U$	98
55	(2,5,6,7)	$D_9 \perp D_4 \perp U$	443
56	(5,6,8,11)	$E_8^2 \perp A_1 \perp U$	2
57	(4,5,6,9)	$E_8 \perp D_5 \perp A_2 \perp U$	36
58	(1,4,5,6)	$T_{2,5,6}$	3344
59	(1,5,7,8)	$E_8 \perp A_2 \perp U$	3045
60	(1,4,6,7)	$E_7 \perp A_2 \perp U$	3343
61	(4,6,7,11)	$E_8 \perp D_8 \perp U$	9
62	(3,4,5,8)	$D_9 \perp D_5 \perp U$	242
63	(1,2,3,4)	$M_{(1,1,2,3),(1,1,1,1),-2}$	4029
64	(3,4,7,10)	$E_8 \perp D_7 \perp U$	230
65	(3,5,11,14)	$E_8^2 \perp U$	88
66	(1,1,2,3)	$M_{(1,2),(1,1),0}$	4302
67	(2,3,7,9)	$E_8 \perp (A_2)^2 \perp U$	1566
68	(3,4,10,13)	$E_8 \perp E_7 \perp U$	221

No.	weights	Picard Lattice	index(Δ)
69	(2,3,4,7)	$D_4 \perp A_7 \perp U$	1555
70	(2,3,5,8)	$E_8 \perp A_2 \perp (A_1)^2 \perp U$	1569
71	(1,3,4,7)	$T_{2,5,5}$	3888
72	(1,2,5,7)	$E_6 \perp U$	4198
73	(7,8,10,25)	$E_8^2 \perp A_1 \perp U$	2
74	(4,5,7,16)	$M_{(3,3,4,6),(1,1,1,3),-4}$	93
75	(2,4,5,11)	$E_7 \perp (A_1)^4 \perp U$	1533
76	(2,5,6,13)	$D_8 \perp D_4 \perp U$	1125
77	(1,5,7,13)	$E_8 \perp A_1 \perp U$	3731
78	(1,4,6,11)	$E_7 \perp A_1 \perp U$	3889
79	(2,5,9,16)	$E_8 \perp D_5 \perp U$	1117
80	(4,5,13,22)	$E_8^2 \perp U$	88
81	(2,3,8,13)	$E_8 \perp (A_1)^3 \perp U$	3261
82	(1,3,7,11)	$E_7 \perp U$	4147
83	(4,5,18,27)	$E_8 \perp E_7 \perp U$	221
84	(5,6,7,9)	$E_8 \perp A_8 \perp U$	6
85	(2,3,4,5)	$D_4 \perp A_6 \perp A_1 \perp U$	1206
86	(4,5,7,9)	$E_8 \perp T_{2,5,5}$	16
87	(1,3,4,5)	$T_{3,4,5}$	3605
88	(2,5,9,11)	$E_8 \perp E_6 \perp U$	473
89	(1,2,3,5)	$M_{(1,2,4),(1,1,2),-2}$	4127
90	(4,6,7,17)	$E_8 \perp D_6 \perp A_1 \perp U$	32
91	(5,6,8,19)	$E_8 \perp E_7 \perp A_1 \perp U$	11
92	(3,5,11,19)	$E_8 \perp E_7 \perp U$	221
93	(3,4,10,17)	$E_8 \perp D_6 \perp U$	476
94	(3,4,5,7)	$M_{(2,3,4,6),(1,1,2,2),-4}$	158
95	(2,3,5,7)	$M_{(1,2,4,6),(1,1,2,3),-4}$	1328

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